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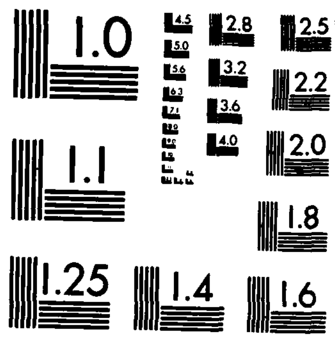

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IN SIMPLE AND COMPOUND WAVEGUIDES

C. H. Wilcox

Technical Summary Report #45

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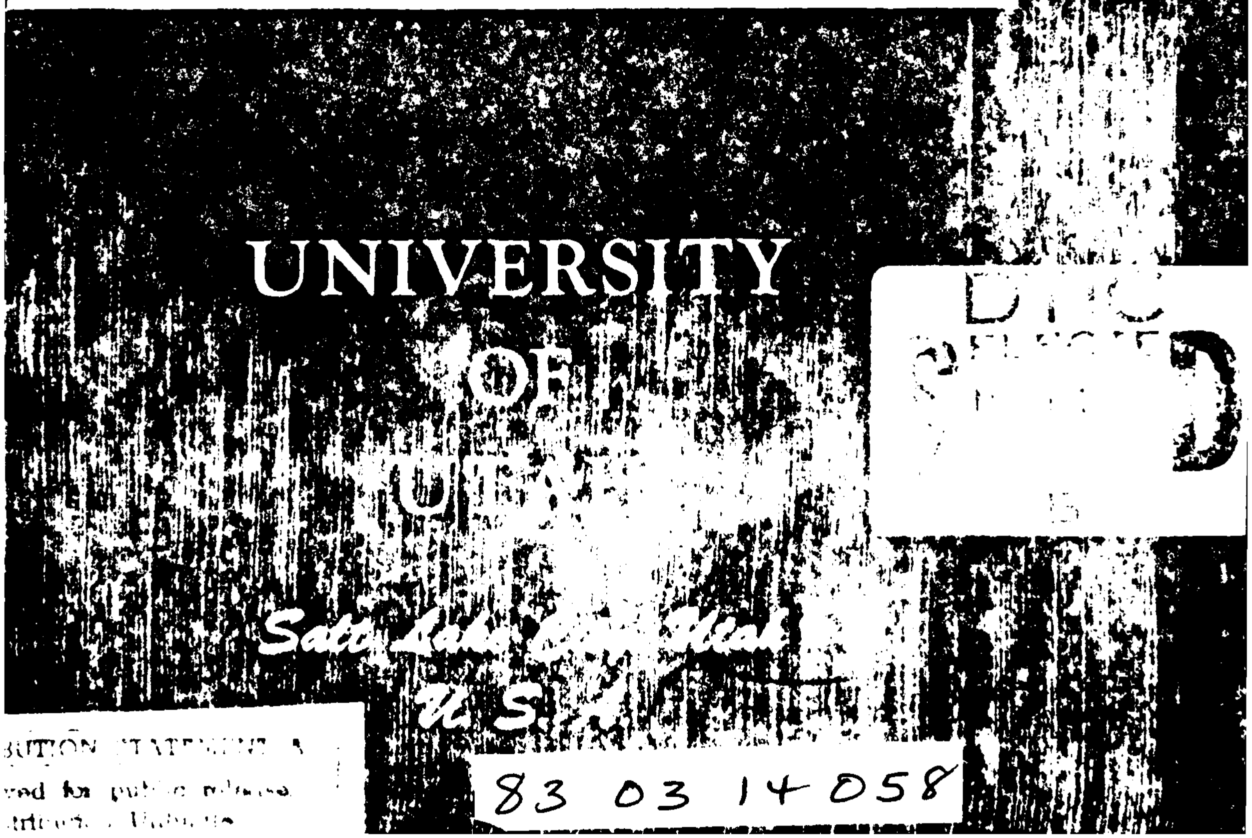

THE S-MATRIX AND ACOUSTIC SIGNAL STRUCTURE

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Abstract

Transient acoustic signals are studied in compound waveguides consisting of a resonant cavity attached to a semi-infinite cylindrical waveguide. The signals are shown to have the asymptotic form

$$v(t,x,y) \sim \sum_{j=0}^{\infty} v_j(t,y)\phi_j(x) \ , \ t \to \infty \ ,$$

where $x = (x_1, x_2)$ are coordinates in the cylinder cross-section, y is a coordinate along the cylinder and t is a time coordinate. Here $\phi_j(x)$ is an eigenfunction for the cylinder cross-section, with eigenvalue μ_j , and

$$v_j(t,y) = \theta(t,y,\mu_j) \ F_j(\mu_j^{\frac{1}{2}} \ y/(t^2 - y^2)^{\frac{1}{2}})$$

where $\theta(t,y,\mu)$ is a universal factor and $F_j(p)$ characterizes the momentum distribution of mode j . It is shown that if both the signal sources and observation point are far from the resonator then

$$F_j(p) = F_j^{dir}(p) + F_j^{sc}(p)$$

where F_j^{dir} is the direct wave that would exist if no resonator were present and

$$F_j^{sc}(p) = \hat{S}F_j^{dir}(p) = \sum_{p^2+\mu_m-\mu_n>0} \overline{S}_{mn}(p)F_j^{dir}(\sqrt{p^2+\mu_n-\mu_m}) \ .$$

$\hat{S}$ is the S-matrix for the compound waveguide and may be calculated from the model functions.

1. Introduction.

This paper deals with the propagation of transient acoustic fields in waveguides that consist of a semi-infinite cylinder coupled to a resonant cavity or "resonator;", ~~see Figure 1.~~

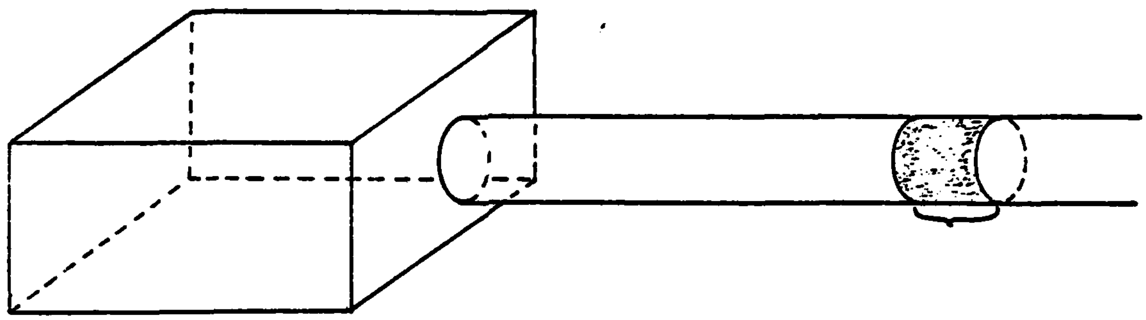

Figure 1. Cylindrical waveguide coupled to a resonator.

The walls of the waveguide are assumed to be rigid. The sources of the transient sound fields, or "signals," are assumed to be localized in a bounded portion of the waveguide and to act for a finite interval of time. The goal of the work is to calculate such acoustic signals and to analyze how their structure depends on the sources and the geometry of the waveguide. Particular attention is given to the cases in which the signal sources or observation point, or both, lie in the cylinder and are far from the resonator.

It is well known that in the cylindrical portion of the waveguide each signal can be decomposed into a series of modal waves. These waves are calculated below and are shown to be asymptotically independent for large times. Moreover, the form of the modal waves is shown to be

determined by the geometry of the cylinder. Only the fine structure of the modal wave profiles varies with the sources and the geometry of the resonator.

The theory of the waveguides of Figure 1 will be developed by perturbation theory, beginning with the special case of the simple waveguide consisting of the semi-infinite cylinder, without resonator, terminated by a plane cap. The general case of a cylinder plus resonator, depicted in Figure 1, will be called a compound waveguide.

Acoustic signal propagation in both simple and compound waveguides will be analyzed by means of normal mode expansions. For simple waveguides the normal modes can be constructed explicitly by separation of variables. For compound waveguides they are constructed by a perturbation method based on those for simple waveguides. The asymptotic wave functions that describe the signals for large times are calculated from the normal mode expansions.

The asymptotic wave functions for a waveguide are characterized by a sequence of functions that, physically, describe the momentum distributions of the modal waves. With each waveguide is associated a scattering operator, or S-matrix, that operates in the space of these momentum distributions. The final result of this paper reveals the significance of the S-matrix in the analysis of acoustic signal structure in waveguides. It states that the momentum distribution of the signal generated in a compound waveguide by sources far from the resonator is simply the image under the S-matrix of the momentum distribution of the signal generated by the same sources in the corresponding simple waveguide.

Normal mode expansions for waveguides and the associated S-matrix were developed by C. Goldstein in the period 1969-74; see [1,2,3,4]*. In 1975 Goldstein's results were extended by W. C. Lyford [5], using results of the abstract theory of scattering. Lyford also presented results on asymptotic wave functions for waveguides in [6]. In 1977 the author presented in [11] an exposition of the theory of normal mode expansions and asymptotic wave functions for the more general case of several semi-infinite cylindrical waveguides coupled by a resonator; see Figure 2.

The purpose of this paper is to present a new construction of the S-matrix, based on the theory of asymptotic wave functions, and to apply the S-matrix to the construction of asymptotic wave functions due

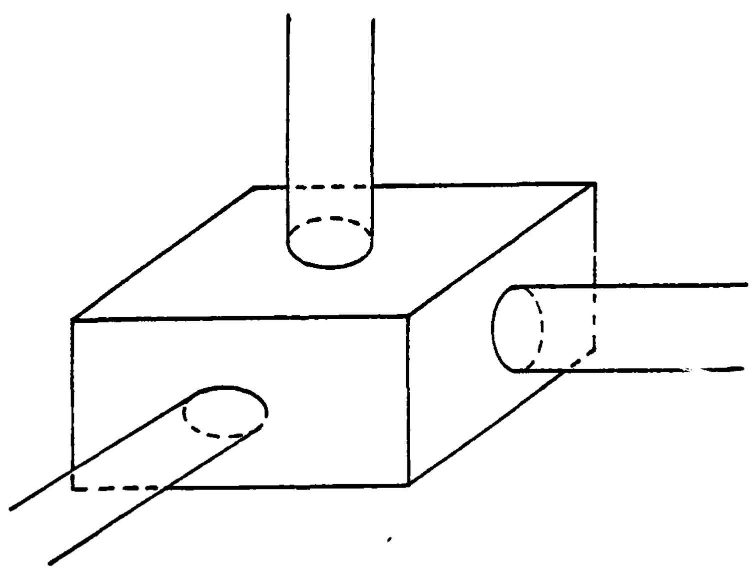

Figure 2. Compound waveguide with resonator and three cylinders.

*Numbers in square brackets indicate references from the list at the end of the paper.

to sources remote from the resonator. The work is based on the earlier literature, primarily the exposition [11], and the author's monograph on the closely related theory of scattering by diffraction gratings [12]. It will be seen that the results presented below can also be derived for acoustically soft boundaries (Dirichlet condition) and elastic boundaries (Robin condition), and for the general case of a resonator with several cylinders. However, to simplify the exposition only the case of a single cylinder and rigid boundary is treated here.

The remainder of the paper is organized as follows. §2 presents a formulation of the propagation problem for acoustic signals in compound waveguides as an initial-boundary value problem for the wave equation, together with its solution by a simple Hilbert space method. §3 develops the normal mode expansion for simple waveguides. In §4 the normal mode functions for compound waveguides are defined and the normal mode expansions for this case are presented. In §5 the normal mode expansions of acoustic signals in waveguides are derived and their asymptotic wave functions, for large times, are calculated. Sections 2 through 5 present a review of concepts and results from [11]. The exposition is therefore concise and without proofs. The new results of this paper are contained in §6 and §7. In §6 the S-matrix of a compound waveguide is defined and then constructed by means of the asymptotic wave functions of §5. In §7 the acoustic signals generated by prescribed sources in simple and compound waveguides are compared. The principal result of this section states that, for sources far from the resonator, the momentum distribution of the scattered signal is simply the image under the S-matrix of the momentum distribution generated by the same sources in the simple waveguide.

2. The Propagation Problem and Its Solution.

The notation that is used in the remainder of the paper is fixed inthis section. Acoustic fields in waveguides will be described by real valued acoustic pressure functions $u(t,X)$ where $t \in R$ is a time coordinate and $X = (x,y) = (x_1,x_2,y) \in R^3$ represents a triple of rectangular coordinates in space. The coordinate axes are assumed to be chosen in such a way that the simple waveguide occupies the semi-infinite cylinder

$$(2.1) \qquad \Omega_0 = G \times R_0 = \{X : x \in G \text{ and } y > 0\}$$

where G is a bounded domain in the x_1,x_2-plane and $R_0 = \{y : y > 0\}$. The corresponding compound waveguide domains are

$$(2.2) \qquad \Omega = \Omega_0 \cup K$$

where K is a bounded domain and Ω is connected, and hence is a domain. The boundary of Ω is denoted by $\partial\Omega$. It must be mentioned that the local structure of $\partial\Omega$ is not completely arbitrary since Ω will be required to have the local compactness property of [11, p. 408]. A simple geometrical property that is sufficient to guarantee this is the finite tiling property of [10]. All the simple piecewise smooth boundaries that arise in applications, such as unions of polyhedra, and finite sections of cylinders, cones, spheres and disks, may be shown to have this property.

The acoustic pressure $u(t,X)$ is the solution of an initial-boundary problem for the wave equation which, in its classical formulation, reads

$$(2.3) \qquad \frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \text{ for } t > 0,\ X \in \Omega,$$

(2.4) $$\frac{\partial u}{\partial \nu} \equiv \vec{\nu} \cdot \nabla u = 0 \text{ for } t \geq 0,\ X \in \partial\Omega,$$

(2.5) $$u(0,X) = f(X) \text{ and } \frac{\partial u(0,X)}{\partial t} = g(X) \text{ for } X \in \Omega$$

where ∇ and Δ are the gradient and Laplace operators in R^3, respectively, while $\vec{\nu}$ is a unit normal field on $\partial\Omega$. The functions $f(X)$ and $g(X)$ in (2.5) characterize the initial state of the acoustic field. They are assumed to be given or calculated from the prescribed wave sources in Ω.

A general theory of the initial-boundary value problem (2.3) - (2.5), guaranteeing the existence and uniqueness of the solution for arbitrary domains, was given in [9]. Of course, for arbitrary domains the boundary condition (2.4) must be understood in a generalized sense. Here a simple alternative to the method of [9] will be based on the acoustic propagator A in the Hilbert space $\mathcal{H} = L_2(\Omega)$. A is the selfadjoint realization of the negative Laplacian defined by

(2.6) $$D(A) = \mathcal{H} \cap \{u : \nabla u \text{ and } \Delta u \text{ are in } \mathcal{H},\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}$$

and

(2.7) $$Au = -\Delta u \text{ for all } u \in D(A).$$

The differential operators ∇ and Δ in (2.6) are to be interpreted in the sense of distribution theory. Of course, if $\partial\Omega$ is not smooth then the boundary condition in (2.6) is to be interpreted in the generalized sense of [9]. With these conventions it can be shown that

(2.8) $$A^* = A \geq 0 \text{ in } \mathcal{H};$$

see [10,11] for details.

With the above definitions the problem (2.3) - (2.5) may be reformulated as an initial value problem for a function $t \to u(t,\cdot) \in \mathcal{H}$; namely

$$\frac{d^2u}{dt^2} + Au = 0 \text{ for } t > 0 \tag{2.9}$$

$$u(0) = f \text{ and } \frac{du(0)}{dt} = g. \tag{2.10}$$

A formal solution is given by

$$u(t,\cdot) = (\cos t\, A^{1/2})\, f + (A^{-1/2} \sin t\, A^{1/2})\, g. \tag{2.11}$$

The spectral theorem implies that (2.11) defines the "solution in $\mathcal{H}$" of (2.9), (2.10) for all $f,g \in \mathcal{H}$. It is also the unique "solution with finite energy" of [9] for all initial states (f,g) with finite energy. This is equivalent to $f \in D(A^{1/2})$, $g \in \mathcal{H}$.

It will be convenient to represent the solution $u(t,X)$ as

$$u(t,X) = \text{Re}\,\{v(t,X)\} \tag{2.12}$$

where $v(t,X)$ is the complex-valued function defined by

$$v(t,\cdot) = e^{-itA^{1/2}}\, h \tag{2.13}$$

and

$$h = f + i\, A^{-1/2}\, g \tag{2.14}$$

This representation is valid if f and g are real valued and f, g and $A^{-1/2}\, g$ are in $\mathcal{H}$. It is sufficient to consider such initial states since the spectral theorem implies that they are dense in $\mathcal{H}$.

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3. Normal Mode Expansions for Simple Waveguides.

The acoustic propagator for the simple waveguide $\Omega_0 = G \times R_0$ will be denoted by A_0. It is selfadjoint and non-negative in the Hilbert space $\mathcal{H}_0 = L_2(\Omega_0)$. The spectral family of A_0 was calculated in [11] and shown to be absolutely continuous. Here the generalized eigenfunction, or normal mode, expansion for A_0 is reviewed briefly.

The normal mode functions for A_0 may be found by separation of variables [11, p. 418ff]. They are defined by

$$(3.1)\quad \psi_j^0(x,y,p) = \left(\frac{2}{\pi}\right)^{1/2} \phi_j(x) \cos py, \quad p > 0,\ j = 0, 1, 2, \cdots ,$$

where the functions

$$(3.2)\qquad \phi_0(x) = |G|^{-1/2},\ \phi_1(x),\ \phi_2(x), \cdots$$

are the eigenfunctions of two-dimensional negative Laplacian $A_G^N = -\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2$ in $L_2(G)$, with Neumann boundary condition. The corresponding eigenvalues of A_G^N, repeated according to their multiplicity will be denoted by

$$(3.3)\qquad \mu_0 = 0 < \mu_1 \le \mu_2 \le \cdots$$

It is known that each μ_j has finite multiplicity and $\mu_j \to \infty$ when $j \to \infty$. Moreover, the smallest eigenvalue is simple and $\phi_0(x) = \text{const.}$ $|G|$ is the 2-dimensional Lebesgue measure of G. The eigenvalue problem for the ψ_j^0, in classical form, is

$$(3.4)\qquad (\Delta + \omega_j^2(p))\ \psi_j^0(X,p) = 0,\ X \in \Omega_0,$$

$$(3.5) \qquad \partial\psi_j^0(X,p)/\partial\nu = 0,\ X \in \partial\Omega_0,$$

where

$$(3.6) \qquad \omega_j(p) = (p^2 + \mu_j)^{1/2} \geq \mu_j^{1/2},\ j = 0,\ 1,\ 2,\cdots .$$

The normal mode expansion theorem for A_0 states that every $h_0 \in \mathcal{H}_0$ can be written as

$$(3.7) \qquad h_0(x,y) = \text{l.i.m.} \sum_{j=0}^{\infty} h_{0j}(y)\ \phi_j(x)$$

where l.i.m. denote convergence in $\mathcal{H}_0$ [11]. The functions $h_{0j} \in L_2(R_0)$ are defined by

$$(3.8) \qquad h_{0j}(y) = \text{l.i.m.} \int_0^\infty \left(\frac{2}{\pi}\right)^{1/2} \cos py\ \hat{h}_{0j}(p)\ dp\ ,$$

where

$$(3.9) \qquad \hat{h}_{0j}(p) = \text{l.i.m.} \int_{\Omega_0} \overline{\psi_j^0(X,p)}\ h_0(X)\ dX\ .$$

In (3.8) and (3.9), l.i.m. denotes convergence in $L_2(R_0)$. The Parseval relation for the expansion (3.7) - (3.9) is contained in the theorem that the linear mapping Φ defined by

$$(3.10) \qquad \Phi h_0 = \{\hat{h}_{00}, \hat{h}_{01}, \hat{h}_{02}, \cdots\}$$

defines a unitary operator

$$\Phi : \mathcal{H}_0 \to \hat{\mathcal{H}}_0 = \sum_{j=0}^{\infty} \oplus L_2(R_0). \tag{3.11}$$

The last notation means that $\hat{\mathcal{H}}_0$ is the direct sum of a countable sequence of copies of $L_2(R_0)$.

The representation Φ is a spectral mapping for A_0. More precisely, one has

$$v_0(t,X) = e^{-itA_0^{1/2}} h_0(X) = \text{l.i.m.} \sum_{j=0}^{\infty} v_{0j}(t,y)\ \phi_j(x) \text{ in } \mathcal{H}_0 \tag{3.12}$$

where

$$v_{0j}(t,y) = \text{l.i.m.} \int_0^{\infty} \left(\frac{2}{\pi}\right)^{1/2} \cos p\, y\, e^{-it\omega_j(p)}\ \hat{h}_{0j}(p)\, dp \tag{3.13}$$

in $L_2(R_0)$.

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4. Normal Mode Expansions for Compound Waveguides.

For compound waveguides $\Omega = K \cup \Omega_0$ the propagator A has two distinct families of normal mode functions. They are denoted here by $\{\psi_j^+(X,p) : p > 0 \text{ and } j = 0, 1, 2, \cdots\}$ and $\{\psi_j^-(X,p) : p > 0$ and $j = 0, 1, 2, \cdots\}$. Both families satisfy the differential equation and boundary condition:

$$(\Delta + \omega_j^2(p))\ \psi_j^\pm(X,p) = 0 \text{ for } X \in \Omega \ , \tag{4.1}$$

$$\partial\psi_j^\pm(X,p)/\partial\nu = 0 \text{ for } X \in \partial\Omega \ . \tag{4.2}$$

The families are distinguished by the condition that in Ω_0 the fields

$$\psi_j^{\pm sc}(X,p) = \psi_j^\pm(X,p) - \psi_j^0(X,p),\ X \in \Omega_0 \ , \tag{4.3}$$

represent outgoing waves for ψ_j^+ and incoming waves for ψ_j^-. This is defined by the condition that the developments of $\psi_j^{\pm sc}$ in the transverse eigenfunctions $\{\phi_m(x)\}$ have the forms

$$\begin{aligned}\psi_j^{\pm sc}(X,p) = {} & \frac{1}{(2\pi)^{1/2}} \sum_{\omega_j^2(p)>\mu_m} T_{jm}^\pm(p)\ e^{\pm iy\sqrt{\omega_j^2(p)-\mu_m}}\ \phi_m(x) \\ & + \frac{1}{(2\pi)^{1/2}} \sum_{\omega_j^2(p)\le\mu_m} T_{jm}^\pm(p)\ e^{-y\sqrt{\mu_m-\omega_j^2(p)}}\ \phi_m(x) \ .\end{aligned} \tag{4.4}$$

The two summations in (4.4) are over the sets of integers $m \geq 0$ for which the indicated inequalities hold. Note that the first sum is finite because $\mu_m \to \infty$ when $m \to \infty$. The coefficients $\{T_{jm}^\pm(p) : \omega_j^2(p) > \mu_m\}$ will be called the scattering amplitudes. It will be shown that they determine the S-matrix for the compound waveguide.

F. Rellich [8] was the first to show that for compound waveguides $\Omega = K \cup \Omega_0$ the acoustic propagator A may have point spectrum $\{\lambda_{(n)} : 1 \leq n < M \leq +\infty\}$ and eigenfunctions

$$\psi_{(n)} \in D(A) \subset \mathcal{H} \tag{4.5}$$

such that

$$A\psi_{(n)} = \lambda_{(n)} \psi_{(n)} \tag{4.6}$$

and

$$\psi_{(n)}(X) = \sum_{\mu_m > \lambda_{(n)}} c_{nm} e^{-y\sqrt{\mu_m - \lambda_{(n)}}} \phi_m(x) . \tag{4.7}$$

The wave functions $\psi_{(n)}$ may be called trapping modes. The corresponding acoustic fields

$$v(t,X) = \sum_{n=1}^{M} c_{(n)} e^{-it\lambda_{(n)}^{1/2}} \psi_{(n)}(X) \tag{4.8}$$

represent standing waves in Ω_0, by (4.7). In particular, the energy of the wave function (4.8) does not propagate to ∞ and hence plays no role in the scattering theory for A. In the remainder of this paper, to simplify the notation, it is assumed that A has no trapping modes. In the general case the results derived below hold in the space of states orthogonal to the trapping modes.

For propagators A with no trapping modes the normal mode expansion theorem states that every $h \in \mathcal{H}$ can be written as

$$h(X) = \text{l.i.m.} \sum_{j=0}^{\infty} h_j^{\pm}(X) \tag{4.9}$$

where the series converges in $\mathcal{H}$. The components $h_j^{\pm} \in \mathcal{H}$ are defined by

$$(4.10) \qquad h_j^{\pm}(X) = \text{l.i.m.} \int_0^{\infty} \psi_j^{\pm}(X,p)\, \hat{h}_j^{\pm}(p)\, dp \ ,$$

with convergence in $\mathcal{H}$, where

$$(4.11) \qquad \hat{h}_j^{\pm}(p) = \text{l.i.m.} \int_{\Omega} \overline{\psi_j^{\pm}(X,p)}\, h(X)\, dX$$

in $L_2(R_0)$. The orthogonality and completeness of the expansions are expressed by the theorem that the linear operators Φ^+ and Φ^- defined by

$$(4.12) \qquad \Phi^{\pm} h = \{\hat{h}_0^{\pm}, \hat{h}_1^{\pm}, \hat{h}_2^{\pm}, \cdots\}$$

are unitary operators from $\mathcal{H}$ to $\hat{\mathcal{H}}_0$. The spectral property of Φ^+ and Φ^- is characterized by the representation

$$(4.13) \qquad v(t,X) = e^{-itA^{1/2}} h(X) = \text{l.i.m.} \sum_{j=0}^{\infty} v_j^{\pm}(t,X) \ ,$$

where

$$(4.14) \qquad v_j^{\pm}(t,X) = \text{l.i.m.} \int_0^{\infty} \psi_j^{\pm}(X,p)\, e^{-it\omega_j(p)}\, \hat{h}_j^{\pm}(p)\, dp \ .$$

The convergence in both (4.13) and (4.14) is in $\mathcal{H}$.

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5. Asymptotic Wave Functions for Waveguides.

The theory of asymptotic wave functions for waveguides, as developed in [6] and [11], is reviewed briefly in this section.

Consider first the wave function $v_0(t,X)$ for the simple waveguide defined by (3.12), (3.13). The modal wave v_{0j} can be written

$$v_{0j}(t,y) = v_{0j}^{out}(t,y) + v_{0j}^{in}(t,y) \tag{5.1}$$

where

$$v_{0j}^{out}(t,y) = v_{0j}^{in}(t,-y) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{i(yp-t\omega_j(p))} \hat{h}_{0j}(p)dp \ . \tag{5.2}$$

Here and in what follows the l.i.m. notation of (3.12), (3.13) is omitted, for brevity, but is always to be understood.

The wave functions of v_{0j}^{out} and v_{0j}^{in} have the form

$$w^{\pm}(t,y,\mu,h) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{i(\pm yp-t\omega(p,\mu))} h(p) \ dp \tag{5.3}$$

where $h \in L_2(R_0)$ and $\omega(p,\mu) = (p^2 + \mu)^{1/2}$. For $\mu > 0$, (5.3) defines a dispersive wave with group speed

$$U(p,\mu) = \frac{\partial\omega(p,\mu)}{\partial p} = \frac{p}{(p^2+\mu)^{1/2}} \tag{5.4}$$

In [11] the method of stationary phase was applied to w^+. The resulting estimate is

$$w^{\infty}(t,y,\mu,h) = \chi\left(\frac{y}{t}\right) \frac{e^{i(yp-t\omega(p,\mu)-\pi/4)}}{(t\partial U(p,\mu)/\partial p)^{1/2}} h(p) \Bigg|_{p = \left(\frac{\mu y^2}{t^2-y^2}\right)^{1/2}}$$

(5.5)

$$= \theta(t,y,\mu)\; h\left(\frac{\mu^{1/2}\, y}{(t^2-y^2)^{1/2}}\right)$$

where

(5.6) $\chi\left(\frac{y}{t}\right)$ = Characteristic function of $\{(t,y) : 0 < \frac{y}{t} < 1\}$

and

(5.7) $\theta(t,y,\mu) = \chi\left(\frac{y}{t}\right) t\mu[\mu(t^2-y^2)]^{-3/4}\; e^{-i[\mu(t^2-y^2)^{1/2}+\pi/4]}$.

For $\mu = 0$, (5.3) reduces to

(5.8) $w^{\pm}(t,y,0,h) = \frac{1}{(2\pi)^{1/2}} \int_0^{\infty} e^{i(\pm y-t)p}\, h(p)\, dp = F^{-1} h(\pm y - t)$

where F is the usual Fourier transform. In this case the waves are non-dispersive and it is appropriate to define

(5.9) $w^{\infty}(t,y,0,h) = F^{-1} h(y-t)$.

With these definitions the results of [11] imply that, for all $h \in L_2(R_0)$ and all $\mu \geq 0$ one has

(5.10) $\lim_{t\to\infty} \| w^{+}(t,\cdot,\mu,h) - w^{\infty}(t,\cdot,\mu,h) \|_{L_2(R_0)} = 0$,

and

(5.11) $\lim_{t\to\infty} \| w^{-}(t,\cdot,\mu,h) \|_{L_2(R_0)} = 0$.

These results are directly applicable to the partial waves v_{0j} defined by (5.1), (5.2). It follows that $v_{0j}^{in}(t,\cdot) \to 0$ and $v_{0j}^{out}(t,\cdot) - v_{0j}^{\infty}(t,\cdot) \to 0$ in $L_2(R_0)$ when $t \to \infty$ where

$$(5.12) \qquad v_{0j}^{\infty}(t,y) = \theta(t,y,\mu_j)\ F_{0j}\left(\frac{\mu_j^{1/2}\, y}{(t^2-y^2)^{1/2}}\right), \ j = 1, 2, \cdots,$$

$$(5.13) \qquad v_{00}^{\infty}(t,y) = (F^{-1}\ F_{00})(y-t)\ ,$$

and

$$(5.14) \qquad F_{0j}(p) = \hat{h}_{0j}(p) \text{ for } j = 0, 1, 2, \cdots .$$

Moreover, it was shown in [11] that if

$$(5.15) \qquad v_0^{\infty}(t,X) = \sum_{j=0}^{\infty} v_{0j}^{\infty}(t,y)\ \phi_j(x)$$

then the series converges in $\mathcal{H}_0$ and

$$(5.16) \qquad \lim_{t\to\infty} \| v_0(t,\cdot) - v_0^{\infty}(t,\cdot)\|_{\mathcal{H}} = 0\ .$$

It is clear from these results that the asymptotic wave function v_0^{∞} is characterized by the sequence $\{F_{0j}\}$ of modal wave momentum distributions or, equivalently, the element

$$(5.17) \qquad F_0 = \{F_{0j}\} = \{\hat{h}_{0j}\} = \Phi h_0 \in \hat{\mathcal{H}}_0\ .$$

The same analysis may be applied to the wave function $v(t,X)$ for the compound waveguide defined by (4.13), (4.14). If the "incoming" normal modes $\{\psi_j^-\}$ are chosen the partial waves have the form

$$(5.18)\qquad v_j^-(t,X) = \int_0^\infty \psi_j^-(X,p)\, e^{-it\omega_j(p)}\, \hat{h}_j^-(p)\, dp\ .$$

For $X \in \Omega_0$ the decomposition (4.3), (4.4) for ψ_j^- may be written

$$(5.19)\qquad \begin{aligned} \psi_j^-(X,p) &= \frac{1}{(2\pi)^{1/2}} e^{ipy}\, \phi_j(x) \\ &+ \frac{1}{(2\pi)^{1/2}} \sum_{m=0}^{\infty} S_{jm}^-(p)\, e^{-iy\sqrt{\omega_j^2(p)-\mu_m}}\, \phi_m(x)\, \chi_{jm}(p) \\ &+ \frac{1}{(2\pi)^{1/2}} \sum_{m=0}^{\infty} S_{jm}^-(p)\, e^{-y\sqrt{\mu_m-\omega_j^2(p)}}\, \phi_m(x)\, \chi'_{jm}(p) \end{aligned}$$

where

$$(5.20)\qquad S_{jm}^-(p) = \delta_{jm} + T_{jm}^-(p)$$

$$(5.21)\qquad \chi_{jm}(p) = \text{Characteristic function of } \{p : \omega_j^2(p) > \mu_m\}$$

and $\chi_{jm}(p) + \chi'_{jm}(p) = 1$. Substituting (5.19) into (5.18) gives

$$(5.22)\qquad v_j^- = v_j^{-,out} + v_j^{-,in} + v_j^{-,\sigma}$$

where the three terms on the right come from the corresponding terms of (5.19).

Clearly

$$(5.23)\qquad v_j^{-,out}(t,X) = w^+(t,y,\mu_j,\hat{h}_j^-)\, \phi_j(x)\ .$$

Moreover,

$v_j^{-,\mathrm{in}}(t,X)$

(5.24)

$$= \sum_{m=0}^{\infty} \left[\frac{1}{(2\pi)^{1/2}} \int_0^{\infty} S_{jm}^{-}(p)\, e^{-iy\sqrt{\omega_j^2(p)-\mu_m} - it\omega_j(p)}\, \hat{h}_j^{-}(p)\, \chi_{jm}(p)\,dp \right] \phi_m(x) .$$

On making the change of variable

$$p'^2 = \omega_j^2(p) - \mu_m = p^2 + \mu_j - \mu_m \tag{5.25}$$

in (5.24) one finds that

$$v_j^{-,\mathrm{in}}(t,X) = \sum_{m=0}^{\infty} w^{-}(t,y,\mu_m,h_{jm}^{-})\, \phi_m(x) \tag{5.26}$$

where the function h_{jm}^{-} is defined by

$$h_{jm}^{-}(p') = \chi_{jm}(p)\, S_{jm}^{-}(p)\, \hat{h}_j^{-}(p)\, \frac{p'}{p} \tag{5.27}$$

and p and p' are related by (5.25). It is clear from (5.11) and (5.26) that, formally,

$$\lim_{t\to\infty} v_j^{-,\mathrm{in}}(t,\cdot) = 0 \text{ in } \mathcal{H}_0 . \tag{5.28}$$

Rigorous proofs of this, and the related property

$$\lim_{t\to\infty} v_j^{-,\sigma}(t,\cdot) = 0 \text{ in } \mathcal{H}_0 , \tag{5.29}$$

were given for the closely related case of diffraction gratings in [12, Part 2]. (5.23), (5.28) and (5.29) imply that

$$v_j^{-}(t,\cdot) - v_j^{-,\infty}(t,\cdot) \to 0 \text{ in } \mathcal{H} \tag{5.30}$$

where

$$(5.31) \qquad v_j^{-,\infty}(t,X) = \chi(X)\, w^{\infty}(t,y,\mu_j,\hat{h}_j^-)\, \phi_j(x)$$

and χ is the characteristic function of Ω_0. Moreover, if

$$(5.32) \qquad v^{-,\infty}(t,X) = \sum_{j=0}^{\infty} v_j^{-,\infty}(t,X)$$

then it follows as in [11] that the series converges in $\mathcal{H}$ and

$$(5.33) \qquad \lim_{t\to\infty} \| v(t,\cdot) - v^{-,\infty}(t,\cdot)\|_{\mathcal{H}} = 0$$

for every $h \in \mathcal{H}$. Note that

$$(5.34) \quad v_j^{-,\infty}(t,X) = \theta(t,y,\mu_j)\, F_j\left(\frac{\mu_j^{1/2}\, y}{(t^2-y^2)^{1/2}}\right) \phi_j(x),\ j = 1, 2, 3, \cdots,$$

$$(5.35) \qquad v_0^{-,\infty}(t,X) = (F^{-1}\, F_0)(y-t)\ |G|^{-1/2}$$

where

$$(5.36) \qquad F_j(p) = \hat{h}_j^-(p) \text{ for } j = 0, 1, 2, \cdots .$$

Thus $v^{-,\infty}$ is characterized by the sequence $\{F_j\}$ of modal momentum distributions where

$$(5.37) \qquad F = \{F_j\} = \{\hat{h}_j^-\} = \Phi^- h \in \hat{\mathcal{H}}_0 .$$

The convergence of $v(t,\cdot)$ to zero in $L_2(K)$, which is implied by (5.33), follows from the local compactness property of [11, p. 408]; see [11, Theorem 6.16].

The results (5.16) and (5.33) imply that each wave function $v(t,\cdot)$ for the compound waveguide is asymptotically equal in $\mathcal{H}$, for $t \to \infty$, to a sum of modal waves for the simple waveguide.

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6. The S-Matrix.

The wave operators $W_\pm = W_\pm(A_0^{1/2}, A^{1/2}, J) : \mathcal{H} \to \mathcal{H}_0$ are defined by

$$(6.1) \qquad W_\pm = \operatorname*{s-lim}_{t \to \pm\infty} e^{itA_0^{1/2}} J\, e^{-itA^{1/2}}$$

where $J : \mathcal{H} \to \mathcal{H}_0$ is the bounded operator defined by $(Jh)(X) = h(X)|_{\Omega_0}$ and s-lim denotes strong convergence. The existence of W_+ follows from (5.16), (5.17), (5.33) and (5.37). More precisely, these results imply that

$$(6.2) \qquad \lim_{t \to \infty} \| e^{itA_0^{1/2}} J\, e^{-itA^{1/2}} h - h_0 \|_{\mathcal{H}_0} = 0$$

provided $F_0 = \Phi h_0 = \Phi^- h = F$. This implies the representation

$$(6.3) \qquad W_+ = \Phi^* \Phi^- .$$

The analogous results for $t \to -\infty$ give

$$(6.4) \qquad W_- = \Phi^* \Phi^+ .$$

These results and the unitarity of Φ and $\Phi^\pm$ imply that $W_\pm$ are unitary operators from $\mathcal{H}$ to $\mathcal{H}_0$.

The scattering operator S for the pair is the unitary operator in $\mathcal{H}_0$ defined by

$$(6.5) \qquad S = W_+ W_-^* .$$

Combining (6.3), (6.4) and (6.5) gives

$$(6.6) \qquad S = \Phi^* \hat{S}\, \Phi$$

where $\hat{S}$ is the unitary operator in the direct sum space $\hat{\mathcal{H}}_0$ of (3.11) that is defined by

$$\hat{S} = \Phi^- \Phi^{+*} . \tag{6.7}$$

$\hat{S}$ will be called the S-matrix for A (or the the waveguide Ω).

The purpose of this section is to calculate an explicit representation of $\hat{S}$ based on the scattering amplitudes $\{S^-_{jm}(p)\}$ of the normal mode functions $\psi^-_j(X,p)$, defined by (4.4) and (5.20). To this end let $h \in \mathcal{H}$ and define elements $\hat{h}^+$ and $\hat{h}^-$ in $\hat{\mathcal{H}}_0$ by (4.12); i.e.,

$$\hat{h}^\pm = \{\hat{h}^\pm_0, \hat{h}^\pm_1, \hat{h}^\pm_2, \cdots\} = \Phi^\pm h . \tag{6.8}$$

Then it follows from the unitarity of Φ^+ and Φ^- that

$$\hat{h}^- = (\Phi^- \Phi^{+*}) \Phi^+ h = \hat{S} \hat{h}^+ . \tag{6.9}$$

This relation will be used to calculate $\hat{S}$. To see how this can be done recall that the calculation of the asymptotic wave function for $v(t,\cdot) = e^{-itA^{1/2}} h$ was based on the incoming representation (5.18) and gave the representation (5.37), i.e., $F = \hat{h}^-$, for the asymptotic momentum distribution. A second calculation of F, based on the outgoing representation, will now be shown to yield a representation of $\hat{S}$.

The outgoing representation of $v(t,\cdot)$ is

$$v(t,X) = \sum_{j=0}^{\infty} v^+_j(t,X) \tag{6.10}$$

where

$$v^+_j(t,X) = \int_0^\infty \psi^+_j(X,p)\, e^{-it\omega_j(p)}\, \hat{h}^+_j(p)\, dp . \tag{6.11}$$

For $X \in \Omega_0$ the decomposition (4.3), (4.4) for ψ_j^+ may be written

$$\psi_j^+(X,p) = \frac{1}{(2\pi)^{1/2}} e^{-ipy} \phi_j(x)$$

(6.12)

$$+ \frac{1}{(2\pi)^{1/2}} \sum_{m=0}^{\infty} S_{jm}^+(p)\, e^{iy\sqrt{\omega_j^2(p)-\mu_m}}\, \phi_m(x)\ \chi_{jm}(p)$$

$$+ \frac{1}{(2\pi)^{1/2}} \sum_{m=0}^{\infty} S_{jm}^+(p)\, e^{-y\sqrt{\mu_m-\omega_j^2(p)}}\, \phi_m(x)\ \chi'_{jm}(p)$$

where

(6.13) $$S_{jm}^+(p) = \delta_{jm} + T_{jm}^+(p) .$$

Substituting (6.12) into (6.11) gives

(6.14) $$v_j^+ = v_j^{+,in} + v_j^{+,out} + v_j^{+,\sigma} ,$$

where the three terms on the right come from the corresponding terms of (6.12). The term

(6.15) $$v_j^{+,in}(t,X) = w^-(t,y,\mu_j,\hat{h}_j^+)\ \phi_j(x)$$

tends to zero in $\mathcal{H}_0$ when $t \to \infty$, by (5.11). Moreover, the method of [12, Part 2, §14] may be used to show that $v_j^{+,\sigma}(t,\cdot) \to 0$ in $\mathcal{H}_0$ when $t \to \infty$.

To calculate the behavior for $t \to \infty$ of $v(t,X)$ from (6.10) - (6.14) it is convenient to begin with the special case of an $h \in \mathcal{H}$ such that

(6.16) $$\hat{h}_j^+(p) = \delta_{jn}\ g(p), \quad n \text{ fixed, and}$$

(6.17) $$\text{supp } g \subset [0,M],\ M \text{ fixed} .$$

For such h,

$$v(t,X) = v_n^+(t,X) = v_n^{+,out}(t,X) + o(1) \tag{6.18}$$

where $o(1) \to 0$ in $\mathcal{H}_0$ when $t \to \infty$. Moreover,

$$\operatorname{supp} \chi_{nm} = [\sqrt{\operatorname{Max}(0,\mu_m-\mu_n)},\infty) \tag{6.19}$$

and hence

$$v_n^{+,out}(t,X) = \sum_{\mu_m \le \mu_n + M^2} \frac{1}{(2\pi)^{1/2}} \int_{\sqrt{\operatorname{Max}(0,\mu_m-\mu_n)}}^{M} S_{nm}^+(p)\, e^{iy\sqrt{\omega_n^2(p)-\mu_m} - it\omega_n(p)}\, \phi_m(x) g(p)\,dp \tag{6.20}$$

The change of variable $p \to p'$ where

$$\omega_n^2(p) = p^2 + \mu_n = p'^2 + \mu_m = \omega_m^2(p') \tag{6.21}$$

gives the representation

$$v_n^{+,out}(t,X) = \sum_{\mu_m \le \mu_n + M^2} \frac{1}{(2\pi)^{1/2}} \int_{\sqrt{\operatorname{Max}(0,\mu_n-\mu_m)}}^{\sqrt{M^2+\mu_n-\mu_m}} S_{nm}^+(p)\, e^{i(yp'-t\omega_m(p'))}\, \phi_m(x)\; g(p)\, \frac{p'}{p}\, dp' \tag{6.22}$$

where in the last integrand $p = \sqrt{p'^2 + \mu_m - \mu_n}$. The definition (5.3) of w^+ implies that (6.22) can be written

$$v_n^{+,out}(t,X) = \sum_{\mu_m \le \mu_n + M^2} w^+(t,y,\mu_m,h_{nm}^+)\, \phi_m(x) \tag{6.23}$$

where

$$h_{nm}^+(p') = S_{nm}^+(p)\; g(p)\, \frac{p'}{p}\, \chi_{mn}(p')\;. \tag{6.24}$$

Note that the lower limit of integration in (6.22), together with (6.19), imply the correctness of the factor $\chi_{mn}(p')$ in (6.24).

Equations (6.18), (6.23) and (5.10) imply that if

$$v^{+,\infty}(t,X) = \sum_{\mu_m \le \mu_n + M^2} w^{\infty}(t,y,\mu_m,h^+_{nm})\ \phi_m(x) \tag{6.25}$$

then

$$\lim_{t\to\infty} \| v(t,\cdot) - v^{+,\infty}(t,\cdot)\|_{\mathcal{H}_0} = 0\ . \tag{6.26}$$

Combining this with (5.33) gives

$$\| v^{-,\infty}(t,\cdot) - v^{+,\infty}(t,\cdot)\|_{\mathcal{H}_0} = 0 \tag{6.27}$$

where

$$v^{-,\infty}(t,X) = \sum_{m=0}^{\infty} w^{+}(t,y,\mu_m,\hat{h}^-_m)\ \phi_m(x)\ . \tag{6.28}$$

Equations (6.25), (6.27) and (6.28) imply that

$$\hat{h}^-_m = h^+_{nm} \text{ for } m = 0,\ 1,\ 2,\cdots\ . \tag{6.29}$$

This is a consequence of a simple lemma which states that if $h = \{h_m\} \in \hat{\mathcal{H}}_0$ and if

$$w(t,X) = \sum_{m=0}^{\infty} w^{\infty}(t,y,\mu_m,h_m)\ \phi_m(x) \to 0 \tag{6.30}$$

in $\mathcal{H}_0$ when $t \to \infty$ then $h = 0$ in $\hat{\mathcal{H}}_0$; i.e., $0 = h_0 = h_1 = h_2 = \cdots$. To verify the lemma note that (6.30) is equivalent to the conditions

$$w^{\infty}(t,y,\mu_m,h_m) = o(1) \text{ in } L_2(R_0),\ m = 0,\ 1,\ 2,\cdots \ . \tag{6.31}$$

To see that (6.31) implies $h_m = 0$ note that, for $\mu > 0$, (5.5) implies

$$\begin{aligned} \|w^{\infty}(t,\cdot,\mu,h)\|^2 &= \int_0^t \left|\mu t[\mu(t^2-y^2)]^{-3/4}\, h\left(\frac{\mu^{1/2}\,y}{(t^2-y^2)^{1/2}}\right)\right|^2 dy \\ &= \int_0^{\infty} |h(p)|^2\, dp \ . \end{aligned} \tag{6.32}$$

The equality of the two integrals follows from the change of variable

$$p = \frac{\mu^{1/2}\,y}{(t^2-y^2)^{1/2}} \ .$$

It is clear from (6.32) that (6.31) implies $h_m = 0$ when $m > 0$. The case $m = 0$ is equally simple; see (5.8).

Equations (6.29) can be written

$$\hat{h}_m^-(p) = \frac{p}{\sqrt{p^2+\mu_m-\mu_n}}\ S_{nm}^+(\sqrt{p^2+\mu_m-\mu_n})\ \hat{h}_n^+(\sqrt{p^2+\mu_m-\mu_n})\ \chi_{mn}(p) \ .$$

These relations were derived for functions $h \in \mathcal{H}$ that satisfy (6.16), (6.17). But sums of such functions are dense in $\mathcal{H}$, by the unitarity of Φ^+ and elementary properties of $\hat{\mathcal{H}}_0$. Hence the general case follows from (6.29) by summing over n and dropping the restriction (6.17). Thus

$$\begin{aligned} \hat{h}_m^-(p) &= \sum_{n=0}^{\infty} \frac{p}{\sqrt{p^2+\mu_m-\mu_n}}\ S_{nm}^+(\sqrt{p^2+\mu_m-\mu_n})\ \hat{h}_n^+(\sqrt{p^2+\mu_m-\mu_n})\ \chi_{mn}(p) \\ &= \sum_{\mu_n<\omega_m^2(p)} \frac{p}{\sqrt{p^2+\mu_m-\mu_n}}\ S_{nm}^+(\sqrt{p^2+\mu_m-\mu_n})\ \hat{h}_n^+(\sqrt{p^2+\mu_m-\mu_n}) \ . \end{aligned} \tag{6.33}$$

It is clear from the last expression that the sum is finite for each fixed $p \geq 0$. Comparison of (6.33) and (6.9) reveals the representation

$$(6.34) \quad (\hat{S}\, h)_m(p) = \sum_{\mu_n < \omega_m^2(p)} \frac{p}{\sqrt{p^2+\mu_m-\mu_n}}\, S_{nm}^+(\sqrt{p^2+\mu_m-\mu_n})\; h_n(\sqrt{p^2+\mu_m-\mu_n})$$

which is valid for all $h = \{h_m\} \in \hat{\mathcal{H}}_0$.

The known unitarity of $\hat{S}$ in $\hat{\mathcal{H}}_0$ can be used to construct an alternative and simpler representation of $\hat{S}$. To derive it note that $\hat{h}^+ = \hat{S}^{-1}\, \hat{h}^-$ and hence an analogue of (6.34) for $\hat{S}^{-1} = \hat{S}^*$ is obtained by interchanging ψ_j^+ and ψ_j^-, and hence $\hat{h}_j^+$ and $\hat{h}_j^-$, in (6.33). The result is

$$(6.35) \quad (\hat{S}^* g)_m(p) = \sum_{\mu_n < \omega_m^2(p)} \frac{p}{\sqrt{p^2+\mu_m-\mu_n}}\, S_{nm}^-(\sqrt{p^2+\mu_m-\mu_n})\; g_n(\sqrt{p^2+\mu_m-\mu_n}) \; .$$

It follows that for all $g,h \in \hat{\mathcal{H}}_0$ one has

$$(g,\hat{S}\, h) = (\hat{S}^* g, h) = \sum_{m=0}^{\infty} \int_0^{\infty} \overline{(\hat{S}^* g)_m(p)}\; h_m(p)\; dp$$

(6.36)

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_0^{\infty} \chi_{mn}(p)\, \frac{p}{\sqrt{p^2+\mu_m-\mu_n}}\, \overline{S_{nm}^-}(\sqrt{p^2+\mu_m-\mu_n})\; \overline{g_n}(\sqrt{p^2+\mu_m-\mu_n})\; h_m(p)\; dp$$

$$= \sum_{n=0}^{\infty} \int_0^{\infty} \overline{g_n}(p') \left(\sum_{m=0}^{\infty} \chi_{nm}(p')\; \overline{S_{nm}^-}(p')\; h_m(\sqrt{p'^2+\mu_n-\mu_m}) \right) dp'$$

where the last step follows from the change of variable $p^2 + \mu_m = p'^2 + \mu_n$. It follows that

$$(\hat{S}\,h)_n(p) = \sum_{m=0}^{\infty} \chi_{nm}(p)\ \overline{S^-_{nm}(p)}\ h_m(\sqrt{p^2+\mu_n-\mu_m})$$

(6.37)

$$= \sum_{\mu_m<\omega_n^2(p)} \overline{S^-_{nm}}(p)\ h_m(\sqrt{p^2+\mu_n-\mu_m})\ .$$

The identities (6.34) and (6.37) clearly imply the relations

$$\frac{p}{\sqrt{p^2+\mu_m-\mu_n}}\, S^+_{nm}(\sqrt{p^2+\mu_m-\mu_n}) = \overline{S^-_{mn}}(p)\ . \tag{6.38}$$

The unitarity of $\hat{S}$ imposes further restrictions on the coefficients $\{S^{\pm}_{nm}\}$. The analogue of (6.37) for $\hat{S}^{-1} = \hat{S}^*$ is obtained from (6.37) by replacing S^-_{mn} by S^+_{mn}. If in (6.37) h is replaced by $\hat{S}^{-1}h$, represented in this way, one finds after simplification using (6.38) that

$$\sum_{\mu_m<\omega_n^2(p)} S^{\pm}_{nm}(p)\ \overline{S^{\pm}_{km}}(\sqrt{p^2+\mu_n-\mu_k})\ \sqrt{p^2+\mu_n-\mu_m} = \sqrt{p^2+\mu_n-\mu_k}\ \delta_{nk}$$

for all $p \geq \sqrt{\mathrm{Max}\ (0,\mu_k-\mu_n)}$.

7. Acoustic Signal Structure in Waveguides.

The signals generated by sources that are localized in a cylindrical portion $\{X = (x,y) : y_0 \le y \le y_0 + \delta_0\}$ of the waveguide Ω are analyzed in this section. To make explicit the dependence on y the initial values will be assumed to have the form

$$\text{(7.1)} \qquad \begin{cases} u(0,X) = f(X,y_0) = f_0(x,y-y_0) \\ \dfrac{\partial u(0,X)}{\partial t} = g(X,y_0) = g_0(x,y-y_0) \end{cases}$$

where

$$\text{(7.2)} \qquad \operatorname{supp} f_0 \cup \operatorname{supp} g_0 \subset G \times [0,\delta_0] \ .$$

For simplicity, the special case for which $g = 0$ will be discussed. The initial state (7.1) with $g = 0$ generates a signal in the simple waveguide Ω_0 whose asymptotic momentum distribution F_0 is given by (5.14) with $h = f(\cdot,y_0)$; i.e.,

$$\text{(7.3)} \qquad F_0(p) = \{F_{0j}(p)\} = \{\hat{f}_j(p,y_0)\}$$

where

$$\text{(7.4)} \qquad \hat{f}_j(p,y_0) = \int_{\Omega_0} \overline{\psi_j^0(X,p)}\, f(X,y_0)\, dX \ .$$

By (3.1),

$$\text{(7.5)} \qquad \psi_j^0(X,p) = \frac{1}{(2\pi)^{1/2}}\, e^{ipy}\, \phi_j(x) + \frac{1}{(2\pi)^{1/2}}\, e^{-ipy}\, \phi_j(x) \ .$$

Substituting this in (7.3), (7.4) gives

$$(7.6) \qquad F_{0j}(p) = F_j^{dir}(p) + F_j^{refl}(p)$$

where

$$(7.7) \qquad F_j^{dir}(p) = F_j^{refl}(-p) = \frac{1}{(2\pi)^{1/2}} \int_{\Omega_0} e^{-ipy}\, \phi_j(x)\, f(X,y_0)\, dX \,.$$

$F_j^{dir}(p)$ and $F_j^{refl}(p)$ represent the momentum distributions generated by $f(\cdot,y_0)$ in a doubly infinite cylinder $G \times R$ which have positive and negative momenta, respectively. Thus, for Ω_0, F_j^{dir} represents the signal component that propagates directly outward while F_j^{refl} represents the component that is reflected at the end $y = 0$ and then propagates outward.

Now consider the signal generated in the compound waveguide $\Omega = \Omega_0 \cup K$ by the same initial state (7.1) with $g = 0$. It has an asymptotic momentum distribution F that is given by (5.36) with $h = f(\cdot,y_0)$; i.e.,

$$(7.8) \qquad F(p) = \{F_j(p)\} = \{f_j^-(p,y_0)\}$$

where

$$(7.9) \qquad \hat{f}_j^-(p,y_0) = \int_{\Omega} \overline{\psi_j^-(X,p)}\, f(X,y_0)\, dX \,.$$

Substituting the decomposition (5.19) into (7.8), (7.9) gives

$$(7.10) \qquad F_j(p) = F_j^{dir}(p) + F_j^{sc}(p)$$

where $F_j^{dir}(p)$ is given by (7.7) and is again interpreted as the direct part of $F_j(p)$. $F_j^{sc}(p)$ may be interpreted as the component of $F_j(p)$ that is due to scattering by the resonator $K = \Omega - \Omega_0$. A short calculation, using (5.19) and (6.37), gives

$$F_j^{sc}(p) = (\hat{S}\, F^{refl})_j(p) + F_j^{\sigma}(p) \tag{7.11}$$

where $F_j^{\sigma}(p)$ contains the exponentially damped terms of (5.19):

$$F_j^{\sigma}(p) = \frac{1}{(2\pi)^{1/2}} \int_{y_0}^{y_0+\delta} \int_G \left\{ \sum_{m=0}^{\infty} \overline{S_{jm}^{-}(p)}\, e^{-y\sqrt{\mu_m - \omega_j^2(p)}}\, \phi_m(x)\, \chi'_{jm}(p) \right\} f(X,y_0)\,dX \, . \tag{7.12}$$

(7.11) and (7.12) suggest that

$$F^{sc} = \hat{S}\, F^{refl} + o(1) \text{ in } \hat{\mathcal{K}}_0,\ y_0 \to \infty; \tag{7.13}$$

that is,

$$\lim_{y_0 \to \infty} \| F^{sc} - \hat{S}\, F^{refl} \|_{\hat{\mathcal{K}}_0} = 0 \, . \tag{7.14}$$

It is not difficult to show from (7.12) that $F_j^{\sigma}(p) \to 0$ when $y_0 \to \infty$, uniformly for p on compact subsets of the set $R - \{\pm\sqrt{\mu_m - \mu_j} : m,j = 0,1,2,\cdots\}$; cf. [12, Lemma 14.1]. However, this is not strong enough to prove (7.14). Unfortunately, (7.14) cannot be proved without additional information about $\psi_j^{-}(X,p)$. This problem was studied in [12] for the analogous case of diffraction gratings. There it was shown [12, Theorem 15.1] that if the analytic continuation of the resolvent $(A - z)^{-1}$ across the spectrum of A has no singularities at the points $z = \mu_m$ then (7.14) holds. The same method can be applied to the waveguide problem of this paper. The method also extends to the case of general initial values (7.1), (7.2).

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Acoustic waveguide
S-matrix
Asymptotic wave function

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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20. ABSTRACT:

Transient acoustic signals are studied in compound waveguides consisting of a resonant cavity attached to a semi-infinite cylindrical waveguide. The signals are shown to have the asymptotic form

$$v(t,x,y) \sim \sum_{j=0}^{\infty} v_j(t,y)\phi_j(x) \ , \ t \to \infty \ ,$$

where $x = (x_1,x_2)$ are coordinates in the cylinder cross-section, y is a coordinate along the cylinder and t is a time coordinate. Here $\phi_j(x)$ is an eigenfunction for the cylinder cross-section, with eigenvalue μ_j , and

$$v_j(t,y) = \theta(t,y,\mu_j) \ F_j(\mu_j^{1/2} \ y/(t^2 - y^2)^{1/2})$$

where $\theta(t,y,\mu)$ is a universal factor and $F_j(p)$ characterizes the momentum distribution of mode j . It is shown that if both the signal sources and observation point are far from the resonator then

$$F_j(p) = F_j^{dir} \ (p) + F_j^{sc} \ (p)$$

where F_j^{dir} is the direct wave that would exist if no resonator were present and

$$F_j^{sc} \ (p) = \hat{S}F_j^{dir} \ (p) = \sum_{p^2+\mu_m-\mu_n>0} \bar{S}_{mn}(p)F_j^{dir} \ (\sqrt{p^2+\mu_n-\mu_m}) \ .$$

$\hat{S}$ is the S-matrix for the compound waveguide and may be calculated from the model functions.

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